THE THEORY OF FRACTURE OF SOLIDS

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A large number of works has been devoted to the theory of fracture of solids and reference is made to some of these [1-12]. The present paper is concerned with the formulation of a theory of fracture of solids. The theory proposed is formal in character and is limited to the investigation of the simplest phenomological features of solids. It is a development of the ideas of Gvozdev [1].

By the fracture of a solid is meant its disintegration under load. The character of the state of stress should be such that the static as well as the kinematic fracture is possible. Investigation of these aspects of fracture is the subject of its simplest theory.

Following Prandtl's [2] concepts let us consider a brittle fractured body. The assumption of a brittle fractured body is a simplifying one and it allows such properties as elasticity, viscosity, plasticity creep etc. which are exhibited by solids under load, to be disregarded. It is the model of a brittle fractured body which permits to exhibit the simplest fracture properties in their "pure" form.

If one excludes the influence of temperature and of rate of loading, then fracture of a brittle fracturing body will occur when some combination of stresses will reach its limiting value

$$f(\sigma_1, \sigma_2, \sigma_3) = c \qquad (c - \text{const}) \tag{1}$$

where σ_1, σ_2, σ_3 are the principal stresses.

One should note that even under these assumptions, the process of fracture can be accompanied by emission or absorption of heat, and this quantity must be considered in the general balance of the energy state; however, in the simplest case we will assume, that all heat effects can be neglected.

Expression (1) will be called the failure condition. The simplest possible failure condition can to a large degree be achieved assuming a homogeneous body, isotropic at any moment of loading.

The first assumption in the simplest case is that the failure condition is independent of the parameters characterizing change of properties in the initial condition of the material.

The second assumption establishes coincidence of material properties in an arbitrary direction.

The failure condition is interpreted in the space of principal stresses as some surface, which will be called the surface of failure condition. Obvious properties of this surface of failure are that it does not pass through the origin of the coordinate system, and that any straight line through the origin of the coordinate system does not intersect it more than once.

Thus, in formulating a simple theory of fracture, the following assumptions have been made:

(1) Absence of properties of elasticity, viscosity, plasticity, creep etc in the body (model of a brittle fractured body).

(2) Absence of influence of heat effects on fracture (ideal character of fracture).

(3) Homogeneous properties of the body.

(4) Isotropic properties of the body.

(5) Disregard of influence of rate of loading etc.

It appears that relaxation of any of these assumptions 1-5 will lead to generalizations of the considered theory of fracture.

Let us now express some assumptions which are to be the basis for further considerations. First let us assume that the failure surface is not concave. This assumption can be investigated following Drucker [13] and Hill [14].

The second assumption consists of the following: of all the possible fracture conditions for a given group of mechanical properties, the actual condition corresponds to the minimum fracture stresses. We shall interpret the fracture stress as the value of the vector length $\sigma(\sigma_1, \sigma_2, \sigma_3)$, where components σ_1, σ_2, σ_3 satisfy equation (1).

Let us investigate a curve lying at the intersection of the fracture condition surface with the surface $\sigma_1 + \sigma_2 + \sigma_3 = \text{const}$ (Fig. 1).

Obviously the possible non-concave curves for the given lengths of

$OA_1 = OA_2 = OA_3$ lie between triangles $A_1A_2A_3$ and $B_1B_2B_3$. For given lengths OA_i and $OC_1 (i = 1, 2, 3)$ the hexagon $A_1A_1A_2C_2A_3C_3$ will obviously define a curve corresponding to the possible fracture surface, requiring minimum fracture stresses. Thus some non-concave curvilinear pyramid is the required surface of fracture condition.

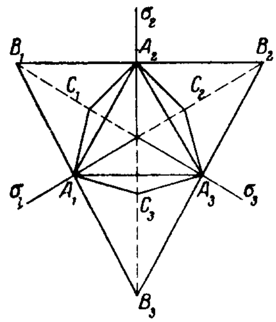

Fig. 1.

It is known [15] that the state of stress corresponding to the edge of such a pyramid, is statically determinate, and the equations defining the state of stress belong, generally speaking, to the hyperbolic type.

This can be proved as follows.

It suffices to investigate the case, when

$$\sigma_1 = \sigma_2, \qquad \sigma_3 = f(\sigma_1) \tag{2}$$

Using relations

$$\sigma_x = \sigma_1 l_1^2 + \sigma_2 m_1^2 + \sigma_3 n_1^2, \ldots \tag{3}$$
$$\tau_{xy} = \sigma_1 l_1 l_2 + \sigma_2 m_1 m_2 + \sigma_3 n_1 n_2, \ldots$$

where σ_x, $\tau_{xy}\ldots$ are stress components, and l_i, m_i, $n_i (i = 1, 2, 3)$ direction cosines, determining the mutual orientation of the axes of the Cartesian system of coordinates and of the principal directions of stress, we obtain from (3) and (2)

$$\begin{aligned} \sigma_x &= \sigma_1 + [f(\sigma_1) - \sigma_1] n_1^2, & \tau_{xy} &= [f(\sigma_1) - \sigma_1] n_1 n_2 \\ \sigma_y &= \sigma_1 + [f(\sigma_1) - \sigma_1] n_2^2, & \tau_{yz} &= [f(\sigma_1) - \sigma_1] n_2 n_3 \\ \sigma_z &= \sigma_1 + [f(\sigma_1) - \sigma_1] n_3^2, & \tau_{zx} &= [f(\sigma_1) - \sigma_1] n_3 n_1 \end{aligned} \tag{4}$$

The value of σ_1 can be expressed through $\sigma = 1/3(\sigma_1 + \sigma_2 + \sigma_3)$ from relation

$$2\sigma_1 + f(\sigma_1) = 3\sigma \tag{5}$$

We will assume $\sigma_1 = g(\sigma)$. From (4) we obtain the required fracture condition

$$\begin{aligned} &[\sigma_x - g(\sigma)][\sigma_y - g(\sigma)] - \tau_{xy}{}^2 = 0 \\ &[\sigma_y - g(\sigma)][\sigma_z - g(\sigma)] - \tau_{yz}{}^2 = 0 \\ &[\sigma_z - g(\sigma)][\sigma_x - g(\sigma)] - \tau_{zx}{}^2 = 0 \end{aligned} \tag{6}$$

Substituting expressions (4) into equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \ldots \tag{7}$$

and considering that

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{8}$$

we obtain a system of four equations with four unknowns, whose characteristic determinant is equal to

$$\Phi\left[\left(2 + \frac{d\varphi}{d\sigma_1}\right)\Phi^2 - (\operatorname{grad}\psi)^2\right] = 0 \tag{9}$$

$$\Phi = \frac{\partial \psi}{\partial x} n_1 + \frac{\partial \psi}{\partial y} n_2 + \frac{\partial \psi}{\partial z} n_3, \qquad \varphi(\sigma_1) = f(\sigma_1) - \sigma_1$$

where $\psi(x, y, z)$ is the equation of the characteristic surface of the system.

From condition (10) it follows that there exist two families of characteristic surfaces, of which one is orthogonal to the direction of the third principal stress, and the second intersects it at an angle θ, so that

$$\cos\theta = \pm \frac{1}{\sqrt{2 + d\varphi/d\sigma_1}} \tag{10}$$

It follows from (10) that

$$df/d\sigma_1 \geqslant -1 \tag{11}$$

Thus if the state of stress preceding failure corresponds to a pyramid edge, then the boundary conditions (formulated of course in terms of stresses) enable us to determine the region in each point of which the fracture conditions (1) are realized. Fracture will take place in this region, which will be termed the fracture region.

Let us investigate some particular cases of fracture conditions. First let us assume that the pyramid interpreting the fracture condition intersects the axis $\sigma_1 = \sigma_2 = \sigma_3$ at a point $\sigma_1 = \sigma_2 = \sigma_3 > 0$. In the opposite case fracture would occur under uniform compression stress. If one assumes that the apex of the pyramid moves towards infinity, then the pyramid degenerates into a cylinder, whose generators are parallel to the straight line $\sigma_1 = \sigma_2 = \sigma_3$. In the simplest case, when the fracture limits in tension and compression coincide, the fracture condition respresents the equality of the maximum shear stress to some constant

$$|\sigma_i - \sigma_j| = c \qquad (i \neq j,\ i,\ j = 1,\ 2,\ 3) \tag{12}$$

Another limiting case occurs when the pyramid apex moves towards the origin of coordinates. If the value of the fracture limit in tension is known and is equal to a constant d, then the pyramid degenerates into a plane

$$\sigma_1 + \sigma_2 + \sigma_3 = d \tag{13}$$

One of the most interesting cases is the fracture condition

$$\sigma_1 \leqslant d, \qquad \sigma_2 \leqslant d, \qquad \sigma_3 \leqslant d, \qquad d = \text{const} \tag{14}$$

Condition (14) is interpreted in the space of principal stresses by a triangular base pyramid, three sides of which are planes meeting at right angles (O_1ABC in Fig. 2). Let us assume that the state of stress corresponds to the pyramid edge

$$\sigma_1 = \sigma_2 = d, \qquad \sigma_3 < d \tag{15}$$

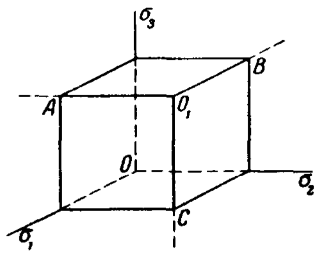

Fig. 2.

In this case we obtain from (3) and (15)

$$\begin{aligned} \sigma_x &= d + qn_1^2, & \tau_{xy} &= qn_1n_2 \\ \sigma_y &= d + qn_2^2, & \tau_{yz} &= qn_2n_3 \\ \sigma_z &= d + qn_3^2, & \tau_{zx} &= qn_3n_1 \end{aligned} \tag{16}$$

where $q = 3(\sigma - d)$.

From (16) we obtain the required fracture conditions

$$\begin{aligned} (\sigma_x - d)(\sigma_y - d) - \tau_{xy}^2 &= 0 \\ (\sigma_y - d)(\sigma_z - d) - \tau_{yz}^2 &= 0 \\ (\sigma_z - d)(\sigma_x - d) - \tau_{zx}^2 &= 0 \end{aligned} \tag{17}$$

Assuming $n_i = \cos\phi_i$ and substituting expression (16) into equation of equilibrium (7), we obtain

$$\cos 2\varphi_1 \frac{\partial q}{\partial x} + \cos\varphi_1 \cos\varphi_2 \frac{\partial q}{\partial y} + \cos\varphi_1 \cos\varphi_3 \frac{\partial q}{\partial z} - q \sin 2\varphi_1 \frac{\partial \varphi_1}{\partial x} - q \sin\varphi_1 \cos\varphi_2 \frac{\partial \varphi_2}{\partial y} -$$

$$- q \sin\varphi_1 \cos\varphi_3 \frac{\partial \varphi_1}{\partial z} - q \cos\varphi_1 \sin\varphi_2 \frac{\partial \varphi_2}{\partial y} - q \cos\varphi_1 \sin\varphi_3 \frac{\partial \varphi_3}{\partial z} = 0, \ldots \tag{18}$$

Joining equation (8) to the three equations (18), we find that the characteristic determinant of the system is equal to

$$\Phi = 0 \tag{19}$$

Consequently, the characteristic surfaces are orthogonal to the directions of the third principal stress. The system of equations is parabolic. Obviously the investigated case is the limiting case of the general one when $df/d\sigma_1 \to \infty$.

Let us pass to the investigation of the kinematic side of the fracture process. It is to be remembered that the body and its parts remain rigid (non-deformable) according to the accepted hypothesis during the whole process of fracture. However, fracture of a given brittle fracturing body under a given state of stress proceeds in a completely defined way.

Let us assume that at each point in the fracture region the fracture velocities are defined. The field of fracture velocities enables us to define the field of fracture strain velocities.

The field of fracture strain velocities is to be understood in the sense that at a moment preceding fracture, the body tends to deform in a fully defined manner. This tendency to deform leads to failure.

The true field of strain velocities must characterize definite extremum properties of state of a body prior to failure. It is therefore natural to consider fracture condition as a "fracture potential".

$$\varepsilon_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{20}$$

Definition (20) allows the formulation of theorems, which establish extremum properties of the state before fracture [16-17].

Generalization of definition (20) shows that it is preferable to satisfy the state of stress, corresponding to the edges of the fracture surface, since this ensures the maximum freedom of fracture [18-19].

Let us indicate relations defining the field of fracture strain velocities in case (6). Following [18] we obtain

$$\begin{gathered}
\varepsilon_x = \frac{1}{3}\lambda_1 g' (\sigma_y + \sigma_z - 2g) + \lambda_2 \left[\frac{1}{3} g' (\sigma_z + \sigma_x - 2g) - \right. \\
\left. - (\sigma_z - g) \right] + \lambda_3 \left[\frac{1}{3} g' (\sigma_x + \sigma_y - 2g) - (\sigma_y - g)\right], \ldots \\
\varepsilon_{xy} = \lambda_3 \tau_{xy}, \qquad \varepsilon_{yz} = \lambda_1 \tau_{yz}, \qquad \varepsilon_{zx} = \lambda_2 \tau_{zx} \qquad (g' = dg/d\sigma)
\end{gathered} \tag{21}$$

From (21) we find

$$\varepsilon_x - \varepsilon_{xy} \frac{{}^1\!/_3 g' (\sigma_x + \sigma_y - 2g) - (\sigma_y - g)}{\tau_{xy}} - \varepsilon_{yz} \frac{{}^1\!/_3 g' (\sigma_y + \sigma_z - 2g)}{\tau_{yz}} - \tag{22}$$

$$-\varepsilon_{zx}\frac{{}^1/_3 g'(\sigma_z+\sigma_x-2g)-(\sigma_z-g)}{\tau_{zx}}=0,\ldots$$

The three relations (22) are equations in three unknown components of fracture velocity u, v, w. The characteristic manifold of equation (22) coincides with the characteristic manifold defined by equation (9).

For the case (17) we will have

$$\begin{aligned}
&\varepsilon_x+\varepsilon_{xy}\frac{\sigma_y-d}{\tau_{xy}}+\varepsilon_{xz}\frac{\sigma_z-d}{\tau_{xz}}=0\\
&\varepsilon_{xy}\frac{\sigma_x-d}{\tau_{xy}}+\varepsilon_y+\varepsilon_{yz}\frac{\sigma_z-d}{\tau_{yz}}=0\\
&\varepsilon_{xz}\frac{\sigma_x-d}{\tau_{xz}}+\varepsilon_{yz}\frac{\sigma_y-d}{\tau_{zy}}+\varepsilon_z=0
\end{aligned}\tag{23}$$

The characteristic manifold of the system of equations (23) is defined from equation (19).

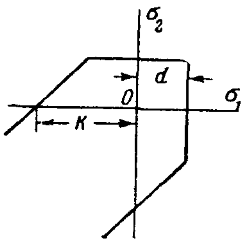

Fig. 3.

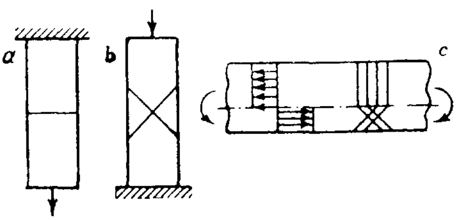

Fig. 4.

As a result of fracture the body disintegrates in a number of rigid parts. Therefore, strictly speaking, only that fracture is feasible whose fracture velocities allow a relative displacement of the parts of the body as rigid entities. Therefore, in brittle fracturing bodies the failure surface will consist of portions of characteristic surfaces (along characteristic surfaces the action of fracture stresses and the tendency to deform are a maximum), so interconnected that part of the body formed during fracture is able to move as a rigid body.

In real bodies under load there appear such properties as elasticity, plasticity etc , therefore, portions of the body have a higher degree of freedom of mutual displacements and fracture is accompanied by the appearance of cracks propagating along characteristic surfaces.

If the failure condition surface is represented by the above mentioned hexagonal curvilinear pyramid, then the failure will be called shear fracture, and failure under condition (15) cleavage. Apparently, under such definition fracture by cleavage is a particular case of fracture in shear.

Some examples follow. Let us investigate the case of the plane state strain. Fracture condition will be expressed as

$$\sigma_1 \leqslant d, \qquad \sigma_2 \leqslant d, \qquad |\sigma_1 - \sigma_2| \leqslant k \tag{24}$$

In Fig. 3 a broken line is shown, representing the fracture condition Note that $\sigma_3 = 1/2(\sigma_1 + \sigma_2)$. Conditions (24) will be rewritten as

$$(\sigma_x - d)(\sigma_y - d) - \tau_{xy}^2 = 0, \qquad (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = k^2 \tag{25}$$

The state of stress under conditions (25) has been investigated in the theory of ideal plastic plane state of stress [20].

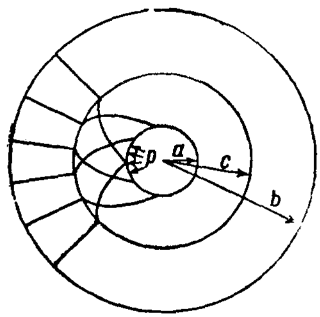

Fig. 5.

In the case of uniaxial tension with $d < k$ cleavage will occur (Fig. 4,a), while in the case of uniaxial compression shear fracture will occur (Fig. 4,b). In the case of pure bending with $d < k$ the position of the neutral axis is obtained from the condition of a minimum bending moment M at fracture. We obtain

$$M = \frac{1}{2}\left[k(H - h)^2 - h^2 d\right] \qquad \left(h = \frac{kH}{k + d}\right) \tag{26}$$

where H is beam thickness, h the distance of the upper side of the beam, from the neutral axis (Fig. 4,c). In the upper part failure occurs as a result of cleavage, in the lower, as a result of shear fracture.

Let us investigate the equilibrium of a thick-walled tube subjected to internal pressure (Fig. 5). We denote the inner and outer tube radii by a and b. One has to distinguish two zones of state in the tube, separated by a circle of radius c. Let us pass to non-dimensional quantities

$$\rho = r/a, \qquad \delta = c/a, \qquad \beta = b/a$$

where r is the current radius.

Using the equation of equilibrium

$$\frac{d\sigma_\rho}{d\rho} + \frac{\sigma_\rho - \sigma_\theta}{\rho} = 0 \tag{27}$$

we obtain from condition $\sigma_\rho - \sigma_\theta = -k$ in the inner zone

$$\sigma_\rho = k \ln \rho + C_1, \qquad \sigma_\theta = k + k \ln \rho + C_1, \qquad \tau_{\rho\theta} = 0, \qquad C_1 = \text{const} \tag{28}$$

In the outer zone

$$\sigma_\rho = \frac{C_2}{\rho} + d, \qquad \sigma_\theta = d, \qquad \tau_{\rho\theta} = 0, \qquad C_2 = \text{const} \tag{29}$$

Combining solutions (28), (29), and taking into consideration that $\sigma_\rho = -p$ at $\rho = 1$, and $\sigma_\rho = 0$ at $\rho = \beta$, we obtain

$$C_1 = -p, \qquad \beta d = \delta k = -C_2 \tag{30}$$

In order to have two zones in the tube, one must satisfy the inequality $d < k$ as $\delta \leqslant \beta$.

The required fracture pressure is obtained from expression

$$p = k - d - \ln \frac{\beta d}{k} \tag{31}$$

In the inner zone failure will occur in shear, in the outer through cleavage.

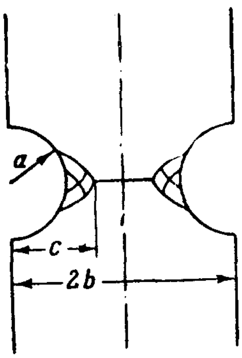

Fig. 6.

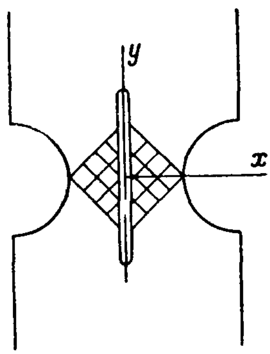

Fig. 7.

Let us investigate a specimen in tension with round notches (Fig. 6). For $d < k$ failure occurs in cleavage. For $d > k$ there appear zones of failure in shear, and the stress is determined from formulas (28), and $\sigma_\rho = 0$ at $\rho = 1$. We then have

$$\sigma_\rho = k \ln \rho, \qquad \sigma_\theta = k + k \ln \rho, \qquad \tau_{\rho\theta} = 0 \tag{32}$$

Relations (29) apply along the cleavage line. Combining these solutions we obtain

$$C_2 = -k\delta, \qquad \delta = \exp\frac{d-k}{k} \qquad (1 \leqslant \delta \leqslant \beta) \tag{33}$$

The value of δ defines the dimensions of the cup formed at failure. When $d \geqslant k + k \ln \beta$ failure occurs in shear only.

If one cuts a vertical slot in the specimen, then the character of the state of stress differs radically from the one investigated, and at $d > k$ failure will occur in shear as shown in Fig. 7.

It is to be noted that, first, there is obviously a similarity between the approach to the study of the fracture problem and the approach to the derivation of a theory of plasticity presented in the paper [21].

However, Saint-Venant [22] has already remarked that plastic deformations and fracture are two radically different phenomena and that he therefore clearly distinguishes between the process of plastic deformation and the process of fracture . Similarity of approach to these problems does not indicate the particular similarity of the processes of plasticity and failure, but indicates the possibility of a unified approach to the problems of investigation of properties of solids, which are exhibited when the load reaches a certain combination of values. These circumstances were illustrated by Prager [23, 24], in his studies of the properties of an ideally hardening and restrictedly compressible body.

Let us also note that such concepts, as discontinuous solutions, statically and kinematically possible fields of stress and fields of velocity, as well as extremum theorems,can be easily adapted from the theory of plasticity to the theory of fracture.

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